RATIONAL REPRESENTATIONS AND PERMUTATION REPRESENTATIONS OF FINITE GROUPS

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Abstract. We investigate the question which $\mathbb{Q}$-valued characters and characters of $\mathbb{Q}$-representations of finite groups are $\mathbb{Z}$-linear combinations of permutation characters. This question is known to reduce to that for quasi-elementary groups, and we give a solution in that case. As one of the applications, we exhibit a family of simple groups with rational representations whose smallest multiple that is a permutation representation can be arbitrarily large.

1. Introduction

Many rational invariants of a finite group G are encoded in the ring $\mathrm{Char}_{\mathbb{Q}}(G)$ of rationally-valued characters, the ring $R_{\mathbb{Q}}(G)$ of rational representations, and the ring $\mathrm{Perm}(G)$ of virtual permutation representations. All three have the same $\mathbb{Z}$-rank, and there are natural inclusions with finite cokernels

$$\mathrm{Perm}(G) \quad \longrightarrow \quad R_{\mathbb{Q}}(G) \quad \longrightarrow \quad \mathrm{Char}_{\mathbb{Q}}(G).$$

The quotient $\mathrm{Char}_{\mathbb{Q}}(G)/R_{\mathbb{Q}}(G)$ is studied by the theory of Schur indices, and the purpose of this paper is to investigate the other two,

$$\mathrm{C}(G) = \frac{R_{\mathbb{Q}}(G)}{\mathrm{Perm}(G)} \qquad \text{and} \qquad \hat{\mathrm{C}}(G) = \frac{\mathrm{Char}_{\mathbb{Q}}(G)}{\mathrm{Perm}(G)}.$$

They have exponent dividing $|G|$ by Artin's induction theorem, and Serre remarked that $\mathrm{C}(G)$ need not be trivial ([14] Exc. 13.4). It is trivial for p-groups [6, 12, 13], and it is known for nilpotent groups [11] (see also §2), for Weyl groups of Lie groups [15, 9] and in other special cases [1, 7]. It follows from the general results of Dress, Kletzing, and Hambleton-Taylor-Williams [4, 5, 9, 8], that the study of $\mathrm{C}(G)$ for a group G reduces, in principle, to that of its quasi-elementary subgroups, or of its 'basic' quasi-elementary subquotients. Specifically, for subgroups the statement is that of the two maps

$$\prod_{\substack{Q\le G\\ \text{quasi-elem.}}} \mathrm{C}(Q) \quad \xrightarrow{\mathrm{Ind}} \quad \mathrm{C}(G) \quad \xrightarrow{\mathrm{Res}} \quad \prod_{\substack{Q\le G\\ \text{quasi-elem.}}} \mathrm{C}(Q),$$

the first one is surjective and the second one injective, and similarly for $\hat{\mathrm{C}}$. This is also an immediate consequence of Solomon's induction theorem, see §3.

Our first observation is that the composite map allows us to describe $\mathrm{C}(G)$ and $\hat{\mathrm{C}}(G)$ explicitly, in a way that bypasses the representation theory of G — purely in terms of quasi-elementary subgroups and the '$\mathrm{Res}\,\mathrm{Ind}$' maps between them; in fact, it is enough to consider maximal quasi-elementary subgroups, i.e. p-normalisers of cyclic subgroups of G. In §3 we give a simple formula for the $\mathrm{Res}\,\mathrm{Ind}$ maps, and in §4 we prove one of the main results of the paper, which describes $\mathrm{C}(Q)$ and $\hat{\mathrm{C}}(Q)$ for a p-quasi-elementary group $Q = C \rtimes P$.

Its simplest formulation is:

Theorem 1.1 (=Theorem 4.6)**.** *Let ρ be an irreducible rational representation of a p-quasi-elementary group $Q = C \rtimes P$. (So C is cyclic, P a p-group, and $p \nmid |C|$.) The order of ρ in* $\mathrm{C}(Q)$ *is $\frac{\dim\hat{\psi}\dim\hat{\pi}}{\dim\rho}$, where $\hat{\psi}$ is the (unique) rational irreducible constituent of* $\mathrm{Res}_C\,\rho$ *and $\hat{\pi}$ a rational irreducible constituent of* $\mathrm{Res}_P\,\rho$ *of minimal dimension.*

Together with the aforementioned '$\mathrm{Res}\,\mathrm{Ind}$' formula, it gives a way to compute $\mathrm{C}(G)$ and $\hat{\mathrm{C}}(G)$ efficiently in a given finite group G. Incidentally, it also gives an algorithm to find $\mathrm{Perm}(G) \subset R_{\mathbb{Q}}(G)$ without computing the subgroup lattice, which is now implemented in Magma [2]. In §5 and §6 we illustrate applications of this approach to proving both triviality and non-triviality of $\mathrm{C}(G)$, as we shall now describe.

In general, $\mathrm{C}(G)$ remains somewhat mysterious, especially in non-soluble groups. Already Frobenius knew that $\mathrm{C}(A_n)$ is trivial for all n. It was announced by Solomon in [15] that $\mathrm{C}(\mathrm{PSL}_2(\mathbb{F}_q))$ is trivial for all prime powers q. In §5 we explain how this, and the same statement for $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{PGL}_2(\mathbb{F}_q)$, follow from the results of §3 and §4.

There is, to our knowledge, no example in the literature of a simple group with non-trivial $\mathrm{C}(G)$. In §6 we show:

Theorem 1.2 (=Theorem 6.1 and Corollary 6.6)**.** *The exponent of the 2-part of* $\mathrm{C}(G)$ *is unbounded in the families $G = \mathrm{PSL}_k(\mathbb{F}_p)$ and $G = \mathrm{SL}_k(\mathbb{F}_p)$. Moreover,* $\hat{\mathrm{C}}(\mathrm{PSL}_k(\mathbb{F}_p)) \neq \{1\}$ *for all even $k \geq 4$ and all odd primes p.*

Notation. Throughout the paper, G denotes a finite group. We write

$\mathrm{Char}(G)$	=	the character ring G,
$\mathrm{Char}_{\mathbb{Q}}(G)$	=	the ring of $\mathbb{Q}$-valued characters,
$R_{\mathbb{Q}}(G)$	=	the ring of characters of virtual $\mathbb{Q}G$-representations,
$\mathrm{Perm}(G)$	=	the ring of characters of virtual permutation representations,
$\mathrm{C}(G)$	=	$R_{\mathbb{Q}}(G)/\mathrm{Perm}(G)$,
$\hat{\mathrm{C}}(G)$	=	$\mathrm{Char}_{\mathbb{Q}}(G)/\mathrm{Perm}(G)$,
$\mathbb{Q}(\chi)$	=	the field of character values of a complex character χ,
$m(\chi)$	=	the Schur index of an irreducible complex character χ over $\mathbb{Q}(\chi)$.

For a complex character χ of G, define its *trace* and, when χ is irreducible, its *rational hull* as

$$\begin{aligned} \mathrm{Tr}\,\chi &= \sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^{\sigma} && \in \mathrm{Char}_{\mathbb{Q}}(G), \\ \hat{\chi} &= m(\chi)\,\mathrm{Tr}\,\chi && \in R_{\mathbb{Q}}(G). \end{aligned}$$

If χ is irreducible, then $\mathrm{Tr}\,\chi$ is a $\mathbb{Q}$-*irreducible character* and $\hat{\chi}$ is the character of an *irreducible rational representation.* We write

$\mathrm{Irr}(G)$	=	the set of (complex) irreducible characters of G,
$\mathrm{Irr}_{\mathbb{Q}}(G)$	=	the set of $\mathbb{Q}$-irreducible characters of G,
$\mu(\alpha,\beta)$	=	$\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ = multiplicity of α in β, used for characters $\alpha \in \mathrm{Irr}_{\mathbb{Q}}(G), \beta \in \mathrm{Char}_{\mathbb{Q}}(G)$, and also for rational representations α, β with α irreducible.

We write $x \sim y$ for conjugate elements. A p-quasi-elementary group is one of the form $G = C \rtimes P$ with C cyclic, and P a p-group; throughout the paper we adopt the convention that $p \nmid |C|$.

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2. Basic facts

Lemma 2.1. *An inclusion $N \triangleleft G$ induces injections $\mathrm{C}(G/N) \hookrightarrow \mathrm{C}(G)$, $\hat{\mathrm{C}}(G/N) \hookrightarrow \hat{\mathrm{C}}(G)$.*

Proof. Suppose $\bar{\rho}$ is a representation of G/N, which lifts to $\rho \in \operatorname{Perm} G$. Write

$$\rho = \bigoplus \mathbb{C}[G/H_i]^{\oplus n_i}, \qquad n_i \in \mathbb{Z}.$$

For a subgroup $H < G$ recall that $\mathbb{C}[G/H]^N \cong \mathbb{C}[G/NH]$, as G-representations (see e.g. [3], proof of Thm. 2.8(5)). Therefore,

$$\bar{\rho} = \rho^N = \bigoplus \mathbb{C}[G/NH_i]^{\oplus n_i} \in \operatorname{Perm}(G/N),$$

as required. □

Lemma 2.2. *Let ρ be an irreducible rational representation and $\tau \in \operatorname{Irr} G$ its constituent, so $\operatorname{Tr} \tau \in \operatorname{Irr}_{\mathbb{Q}}(G)$ and $\rho = m(\tau) \operatorname{Tr} \tau$. The order of $\operatorname{Tr} \tau$ in $\hat{\mathrm{C}}(G)$ is $m(\tau)$ times the order of ρ in $\mathrm{C}(G)$.*

Proof. Clear from the definitions of $\mathrm{C}(G)$ and $\hat{\mathrm{C}}(G)$. □

This allows us to immediately deduce results about $\hat{\mathrm{C}}(G)$ from those about $\mathrm{C}(G)$, and conversely.

Nilpotent groups. Some statements seem to have a cleaner formulation for $\mathrm{C}(G)$ than for $\hat{\mathrm{C}}(G)$, and for some it is the other way around. Let us briefly illustrate this with an example of nilpotent groups:

Theorem 2.3 (Rasmussen [11] Thm 5.2)**.** *Let $G = G_2 \times G_{2'}$ be a nilpotent group, where G_2 is its Sylow 2-subgroup. Then $\mathrm{C}(G)$ is trivial, unless $G_{2'} \neq \{1\}$ and there exists an irreducible character χ of G_2 with $m(\chi) = 2$ and such that one of the following holds:*

1. *$\mathbb{Q}(\chi) \neq \mathbb{Q}$, or*
2. *$\mathbb{Q}(\chi) = \mathbb{Q}$ and there exists a prime divisor q of $|G|$ such that the order of 2 $\pmod{q}$ is even.*

The conditions turn out to be much simpler if one transforms this into a result about $\hat{\mathrm{C}}(G)$. The following follows easily from [11, Thm. 4.2] and standard facts about Schur indices:

Theorem 2.4. *Let $\chi = \chi_2 \chi_{2'}$ be an irreducible character of a nilpotent group $G = G_2 \times G_{2'}$ as above. Then the order of $\operatorname{Tr} \chi$ in $\hat{\mathrm{C}}(G)$ is $m(\chi_2)$ (which is 1 or 2).*

Metabelian and supersolvable groups. The following theorem will be of central importance in what follows. It implies that knowing the order of every $\mathbb{Q}$-irreducible representation in $\hat{\mathrm{C}}(G)$ determines the structure of $\hat{\mathrm{C}}(G)$ completely when G is metabelian or supersoluble (e.g. nilpotent or quasi-elementary). It does not hold in arbitrary groups, as first noted by Berz [1]; the smallest counterexample is $G = C_3 \times \mathrm{SL}_2(\mathbb{F}_3)$.

Theorem 2.5 (Berz [1])**.** *If G is metabelian or supersoluble, then $\mathrm{Perm}(G) \subseteq R_{\mathbb{Q}}(G)$ is freely generated by $n_\rho \rho$, as ρ ranges over irreducible rational representations of G, and*

$$n_\rho = \gcd_{H \le G} \mu(\rho, \mathbb{Q}[G/H]).$$

Lemma 2.6. *If $G = A \rtimes V$ with A abelian and V an elementary abelian p-group, then $\hat{\mathrm{C}}(G) = \{1\}$.*

Proof. By Theorem 2.5, it is enough to show that every complex irreducible character τ of G occurs exactly once in $\mathbb{C}[G/H]$ for a suitable $H < G$. This is clear when $\dim \tau = 1$. Otherwise $\tau = \mathrm{Ind}_{AU}^G \chi$, for some subgroup U of V and a 1-dimensional character χ of AU (see [14, Part II, §8.2]). Let H be a subgroup of V that is complementary to U, i.e. $HU = V$ and $H \cap U = \{1\}$. By Mackey's formula, we have

$$\langle \tau, \mathbb{C}[G/H] \rangle = \langle \chi, \mathrm{Res}_{AU} \mathrm{Ind}_H^G \mathbf{1} \rangle = \langle \chi, \mathrm{Ind}_{AU \cap H}^{AU} \mathbf{1} \rangle = \langle \chi, \mathbb{C}[AU] \rangle = 1. \qquad \square$$

Recall that a p-quasi-elementary group $G = C \rtimes P$ is *basic* if the kernel K of $P \to \mathrm{Aut}(C)$ is trivial or isomorphic to D_8 or has normal p-rank one.

Proposition 2.7 ([7], Proposition 5.2)**.** *Let $G = C \rtimes P$ be basic p-quasi-elementary. Let A_p be a maximal cyclic subgroup of $K = \ker(P \to \mathrm{Aut}(C))$ that is normal in P (it is all of K if K is cyclic, and has index 2 in K otherwise), let $A = CA_p$, and let χ be a faithful one-dimensional character of A. Then $\rho = \mathrm{Tr}\,\mathrm{Ind}_A^G \chi$ is a $\mathbb{Q}$-irreducible character, and*

$$\text{order of } \rho \text{ in } \hat{\mathrm{C}}(G) = \frac{|P|}{|A_p| \cdot \max\limits_{\substack{H \le P \\ H \cap A_p = 1}} |H|}.$$

3. $\hat{\mathrm{C}}(G)$ as a Mackey functor

Let $\mathcal{R}$ be a Mackey subfunctor of the character ring Mackey functor $\mathrm{Char}(G)$. This simply means that for any finite group G, $\mathcal{R}(G)$ is a subgroup of $\mathrm{Char}(G)$ such that if $H \le G$ are finite groups, then

- for all $\rho \in \mathcal{R}(H)$, $\mathrm{Ind}_H^G \rho \in \mathcal{R}(G)$,
- for all $\tau \in \mathcal{R}(G)$, $\mathrm{Res}_H \tau \in \mathcal{R}(H)$,
- for all $\rho \in \mathcal{R}(H)$ and $g \in G$, $\rho^g \in \mathcal{R}(H^g)$.

Here are some examples:

- $R_K(G)$, the representation ring of G over a fixed subfield K of $\mathbb{C}$,
- $\mathrm{Char}_K(G)$, the ring generated by K-valued characters, with fixed $K \subset \mathbb{C}$,
- $\mathrm{Perm}(G)$, the ring generated by permutation characters,
- the subgroup of $\mathrm{Char}(G)$ generated by characters of degree divisible by a fixed integer n.

If p is a prime number, write $\mathcal{R}(G)_p$ for $\mathcal{R}(G)\otimes\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

Proposition 3.1. *Let G be a finite group, fix a prime number p, and let $\mathcal{F}_p$ be a family of subgroups of G such that every p-quasi-elementary subgroup of G is conjugate to a subgroup of some $Q \in \mathcal{F}_p$. Then*

$$\prod_{Q\in\mathcal{F}_p} \operatorname{Res}_Q : \mathcal{R}(G)_p \longrightarrow \prod_{Q\in\mathcal{F}_p} \mathcal{R}(Q)_p$$

is injective. Dually,

$$\sum_{Q} \operatorname{Ind}_Q^G : \prod_{Q\in\mathcal{F}_p} \mathcal{R}(Q)_p \longrightarrow \mathcal{R}(G)_p$$

is surjective.

Proof. By Solomon's induction theorem, a prime-to-p multiple d of the trivial representation can be written as

$$d\mathbf{1}_G = \sum_i n_i \operatorname{Ind}_{H_i}^G \mathbf{1}_{H_i}$$

for some p-quasi-elementary subgroups H_i and integers n_i. Because $\operatorname{Ind}_{H_i^g}^G \mathbf{1}_{H_i^g} \cong \operatorname{Ind}_{H_i}^G \mathbf{1}_{H_i}$, we may assume that each H_i is contained in some $Q_i \in \mathcal{F}_p$. Taking tensor products with any $\rho \in \mathcal{R}(G)$ yields

$$d\rho = \sum_i n_i \operatorname{Ind}_{H_i}^G \operatorname{Res}_{H_i} \rho.$$

If all $\operatorname{Res}_{H_i} \rho$ were 0, then so would be $d\rho$, and therefore also ρ. This proves injectivity. Also, the equation shows that $d\rho \in \operatorname{Im}\left(\sum_Q \operatorname{Ind}_Q^G \mathcal{R}(Q)\right)$, which proves surjectivity, since d is invertible in $\mathbb{Z}_p$. □

Corollary 3.2. *For $S, T \in \mathcal{F}_p$ write $\alpha_{S,T} = \operatorname{Res}_T \operatorname{Ind}_S^G : \hat{\mathrm{C}}(S) \longrightarrow \hat{\mathrm{C}}(T)$. Then*

$$\hat{\mathrm{C}}(G)_p \cong \mathit{Image}\Big(\prod_T \sum_S \alpha_{S,T} : \prod_{S\in\mathcal{F}_p} \hat{\mathrm{C}}(S) \longrightarrow \prod_{T\in\mathcal{F}_p} \hat{\mathrm{C}}(T)\Big).$$

In particular, $\hat{\mathrm{C}}(G)_p = 1$ if and only if for all pairs $S, T \in \mathcal{F}_p$ and all $\rho \in R_{\mathbb{Q}}(S)$ (equivalently, for those ρ whose class in $\mathrm{C}(S)$ is nontrivial), we have $\operatorname{Res}_T^G \operatorname{Ind}_S^G \rho \in \mathrm{Perm}(T)$. The same also holds for $\mathrm{C}(G)$.

Proof. Apply Proposition 3.1 to $\mathcal{R}$ being $\mathrm{Perm}, R_{\mathbb{Q}}$, and $\mathrm{Char}_{\mathbb{Q}}$. □

Corollary 3.3. *Let $\mathcal{F}$ be a family of subgroups of G such that every quasi-elementary subgroup is conjugate to a subgroup of some $Q \in \mathcal{F}$. Then*

$$\hat{\mathrm{C}}(G) \hookrightarrow \prod_{Q\in\mathcal{F}} \hat{\mathrm{C}}(Q)$$

via the (product of) restriction maps. Consequently, the kernel of the composition

$$R_{\mathbb{Q}}(G) \xrightarrow{\prod \operatorname{Res}} \prod_{Q\in\mathcal{F}} R_{\mathbb{Q}}(Q) \longrightarrow \prod_{Q\in\mathcal{F}} \hat{\mathrm{C}}(Q)$$

is $\mathrm{Perm}(G)$. *Dually, the composition*

$$\prod_{Q\in\mathcal{F}} R_{\mathbb{Q}}(Q) \xrightarrow{\operatorname{Ind}} R_{\mathbb{Q}}(G) \to \hat{\mathrm{C}}(G)$$

is onto. The same holds with $R_{\mathbb{Q}}$ *replaced by* $\mathrm{Char}_{\mathbb{Q}}$ *and* $\hat{C}$ *by* C.

Remark 3.4. The theorem and the two corollaries give a very efficient way of computing $\hat{\mathrm{C}}(G)_p, \hat{\mathrm{C}}(G)$, $\mathrm{C}(G)_p$, $\mathrm{C}(G)$ and of finding $\mathrm{Perm}(G)$ as a subring of $R_{\mathbb{Q}}(G) \leq \mathrm{Char}_{\mathbb{Q}}(G)$, without computing the full lattice of subgroups of G.

Remark 3.5. One possible family $\mathcal{F}_p$ is the set of maximal p-quasi-elementary subgroups of G. These are of the form

$$Q = C \rtimes \mathrm{Syl}_p(N_G(C)),$$

where C is cyclic of order prime to p. Possible families $\mathcal{F}$ in Corollary 3.3 are $\mathcal{F} = \bigcup_p \mathcal{F}_p$, as p ranges over prime divisors of $|G|$, or alternatively $\mathcal{F} = \{N_G(C)\}$ as C ranges over (representatives of conjugacy classes of) cyclic subgroups of G.

Notation 3.6. For the remainder of this section we use the following notation:

$CC(G)$	$=$	the set of conjugacy classes of G,
$CC_{\mathrm{cyc}}(G)$	$=$	the set of conjugacy classes of cyclic subgroups of G,
$[x]$	$=$	the conjugacy class of x, when x is either an element of G or a cyclic subgroup,
$\mathrm{Tr}^* \chi$	$=$	the normalised trace $\mathrm{Tr}^* \chi = \frac{1}{[\mathbb{Q}(\chi):\mathbb{Q}]} \mathrm{Tr}\, \chi$ of a character χ,
$\tau(D)$	$=$	$\tau(y)$, where $D \leq G$ is a cyclic subgroup, y is any generator of D, and $\tau \in \mathrm{Char}_{\mathbb{Q}}(G) \otimes \mathbb{Q}$. The rationality of τ ensures that $\tau(y)$ only depends on D and not on the generator y.

Note in particular, that for any character χ of G and any cyclic subgroup D of G, $\mathrm{Tr}^* \chi(D)$ is the average value of χ on the generators of D.

Lemma 3.7. *Let* H_1, H_2 *be two subgroups of* G. *Let* τ_i *be a character of* H_i, $i = 1, 2$, *and assume that* τ_1 *is* $\mathbb{Q}$*-valued. Then*

$$\langle \mathrm{Ind}_{H_1}^G \tau_1, \mathrm{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{\mathrm{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \mathrm{Tr}^* \tau_2(D_2).$$

Proof. First, note that by definition of inner products and of induced class functions,

$$\langle \mathrm{Ind}_{H_1}^G \tau_1, \mathrm{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \overline{\Big(\sum_{y \in [x] \cap H_1} \tau_1(y) \Big)} \Big(\sum_{y \in [x] \cap H_2} \tau_2(y) \Big).$$

The idea of the proof is to partition the set of conjugacy classes of elements of G according to conjugacy classes of cyclic subgroups they generate, and to use the fact that for a rational character τ, $\tau(x) = \tau(x')$ whenever x and x' generate conjugate cyclic subgroups. We get

$$\begin{aligned} &\langle \mathrm{Ind}_{H_1}^G \tau_1, \mathrm{Ind}_{H_2}^G \tau_2 \rangle \\ &\quad = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \overline{\Big(\sum_{y \in [x] \cap H_1} \tau_1(y) \Big)} \Big(\sum_{y \in [x] \cap H_2} \tau_2(y) \Big) \\ &\quad = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{\mathrm{cyc}}(G)} f(C), \end{aligned}$$

where

$$\begin{aligned}
f(C) &= |Z_G(C)| \cdot \sum_{\substack{[x]\in CC(G)\\ \langle x\rangle = C}} \Big(\sum_{y\in[x]\cap H_1} \tau_1(y)\Big)\Big(\sum_{y\in[x]\cap H_2} \tau_2(y)\Big) \\
&= |Z_G(C)| \cdot \#\{k : x \sim x^k\} \cdot \\
&\quad \cdot \sum_{\substack{D_1\le H_1\\ D_1\sim C}} \tau_1(D_1) \cdot \sum_{\substack{[x]\in CC(G)\\ [x]\sim C}} \sum_{y\in[x]\cap H_2} \tau_2(y) \\
&= |N_G(C)| \cdot \sum_{\substack{D_1\le H_1\\ D_1\sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2\le H_2\\ D_2\sim C}} \sum_{\substack{\text{generators}\\ y \text{ of } D_2}} \tau_2(y) \\
&= |N_G(C)| \cdot \sum_{\substack{D_1\le H_1\\ D_1\sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2\le H_2\\ D_2\sim C}} \phi(|C|)\,\mathrm{Tr}^*_{\mathbb{Q}/\mathbb{Q}}\,\tau_2(y),
\end{aligned}$$

as claimed. □

Corollary 3.8. *Suppose $H_1 < Q_1 < G$, $H_2 < Q_2 < G$, and let χ_i be irreducible characters of H_i. Set $\tau_i = \mathrm{Ind}_{H_i}^{Q_i}\chi_i$, and $\rho_i = \mathrm{Tr}\,\tau_i$. Assume that τ_2 is irreducible. Then*

$$\begin{aligned}
\mu(\rho_2, \mathrm{Res}_{Q_2}\mathrm{Ind}_{Q_1}^G \rho_1) &= \frac{[\mathbb{Q}(\tau_1):\mathbb{Q}]}{|H_1|\cdot|H_2|} \sum_{[C]\in CC_{\mathrm{cyc}}(G)} |N_G(C)|\phi(|C|)\cdot \\
&\quad \cdot \sum_{\substack{D_1\le H_1\\ D_1\sim C}} \mathrm{Tr}^*\chi_1(D_1) \cdot \sum_{\substack{D_2\le H_2\\ D_2\sim C}} \mathrm{Tr}^*\chi_2(D_2).
\end{aligned}$$

Proof.

$$\begin{aligned}
\mu(\rho_2, \mathrm{Res}_{Q_2}\mathrm{Ind}_{Q_1}^G \rho_1) &= \langle \tau_2, \mathrm{Res}_{Q_2}\mathrm{Ind}^G(\textstyle\sum \tau_1^\sigma)\rangle \\
&= \langle \mathrm{Ind}_{H_2}^G \chi_2, \mathrm{Ind}_{H_1}^G(\textstyle\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\tau_1)/\mathbb{Q})}(\chi_1)^\sigma)\rangle \\
&= \tfrac{1}{[\mathbb{Q}(\chi_1):\mathbb{Q}(\tau_1)]}\langle \mathrm{Ind}_{H_2}^G\chi_2, \mathrm{Ind}_{H_1}^G(\mathrm{Tr}\,\chi_1)\rangle \\
&= \tfrac{1}{[\mathbb{Q}(\chi_1):\mathbb{Q}(\tau_1)]|H_1|\cdot|H_2|} \textstyle\sum_{[C]\in CC_{\mathrm{cyc}}(G)} |N_G(C)|\phi(|C|)\cdot \\
&\qquad \cdot \textstyle\sum_{\substack{D_1\le H_1\\ D_1\sim C}} \mathrm{Tr}\,\chi_1(D_1)\cdot \sum_{\substack{D_2\le H_2\\ D_2\sim C}} \mathrm{Tr}^*\chi_2(y) \\
&= \tfrac{[\mathbb{Q}(\tau_1):\mathbb{Q}]}{|H_1|\cdot|H_2|} \textstyle\sum_{[C]\in CC_{\mathrm{cyc}}(G)} |N_G(C)|\phi(|C|)\cdot \\
&\qquad \cdot \textstyle\sum_{\substack{D_1\le H_1\\ D_1\sim C}} \mathrm{Tr}^*\chi_1(D_1)\cdot \sum_{\substack{D_2\le H_2\\ D_2\sim C}} \mathrm{Tr}^*\chi_2(D_2).
\end{aligned}$$

□

Lemma 3.9. *If C is a cyclic group, and χ is a 1-dimensional character of C, then $(\mathrm{Tr}^*\chi)(C) = \mu(\mathrm{ord}(\chi))/\phi(\mathrm{ord}(\chi))$, where μ is the Moebius mu function, and $\mathrm{ord}(\chi)$ is the smallest natural number n such that $\chi^n = \mathbf{1}$.*

Proof. It is enough to prove the lemma for faithful characters χ, since we may, without loss of generality, replace C by $C/\ker\chi$. Let g be a generator of C. Then

$$(\mathrm{Tr}^*\chi)(C) = \frac{1}{[\mathbb{Q}(\chi):\mathbb{Q}])}\mathrm{Tr}\,\chi(g) = \frac{1}{\phi(\mathrm{ord}(\chi))}\mathrm{Tr}\,\chi(g).$$

If $|C| = n$, then $\chi(g)$ is a primitive n-th root of unity, and the fact that its trace is $\mu(n)$ is classical. □

Corollary 3.10. *Let G be a group and p^r a prime power. Then $\hat{\mathrm{C}}(G)$ has an element of order p^r if and only if there exist two p-quasi-elementary subgroups Q_1, Q_2 of G, irreducible monomial characters $\tau_i = \mathrm{Ind}_{H_i}^{Q_i}\chi_i$ of Q_i, and an integer k, such that*

- *the rational character* $\operatorname{Tr}\tau_2$ *has order divisible by* p^{k+r} *in* $\hat{\mathrm{C}}(Q_2)$*, and*
- *the rational number*

$$\frac{[\mathbb{Q}(\tau_1):\mathbb{Q}]}{|H_1||H_2|}\cdot\sum_{[U]\in CC_{\mathrm{cyc}}(G)}|N_G(U)|\phi(|U|)\cdot$$
$$\cdot\sum_{\substack{D_1\le H_1\\ D_1\sim U}}\frac{\mu([D_1:D_1\cap\ker\chi_1])}{\phi([D_1:D_1\cap\ker\chi_1])}\cdot\sum_{\substack{D_2\le H_2\\ D_2\sim U}}\frac{\mu([D_2:D_2\cap\ker\chi_2])}{\phi([D_2:D_2\cap\ker\chi_2])}$$

has p*-adic valuation at most* k*.*

In this case, $\operatorname{Ind}_{Q_1}^{G}\operatorname{Tr}\tau_1$ *has order divisible by* p^r *in* $\hat{\mathrm{C}}(G)$*.*

Remark 3.11.

- Note that it is enough to take the last two sums in the above formula only over those D_i for which $D_i\cap\ker\chi_i$ has square-free index in D_i, since for the others $\mu(\operatorname{ord}(\operatorname{Res}_{D_i}\chi_i))=0$. For example if χ_i are faithful, then the outer sum may be taken over U of square free order.
- If, say, H_1 is cyclic, the sum $\sum_{\substack{D_1\le H_1\\ D_1\sim U}}$ has at most one term for every U.
- If Q_1, Q_2 are basic and H_1, H_2 are cyclic, then Proposition 2.7 gives a simple expression for the order of $\operatorname{Tr}\tau_2$ in $\hat{\mathrm{C}}(Q_2)$.

Proof of Corollary 3.10. By Corollary 3.2, $\hat{\mathrm{C}}(G)_p$ has an element of order p^r if and only if there exist p-quasi-elementary subgroups Q_1, Q_2, and characters $\rho_i\in\operatorname{Irr}_{\mathbb{Q}}(Q_i)$, such that ρ_2 has order p^{k+r} in $\hat{\mathrm{C}}(Q_2)$ for some k, and $\mu(\rho_2,\operatorname{Res}_{Q_2}\operatorname{Ind}^{G}\rho_1)$ has p-adic valuation at most k. Quasi-elementary groups are M-groups, so if τ_i is a complex irreducible constituent of ρ_i, then there exist subgroups $H_i\le Q_i$ such that $\tau_i=\operatorname{Ind}_{H_i}^{Q_i}\chi_i$ for 1-dimensional characters $\chi_i\in\operatorname{Irr}(H_i)$. The result therefore follows from Corollary 3.8 in combination with Lemma 3.9. $\square$

4. Quasi-elementary groups

The aim of this section is to provide several formulae of theoretical and algorithmic interest for the orders of characters in $\hat{\mathrm{C}}(G)$ and $\mathrm{C}(G)$ when G is quasi-elementary. Let $G=C\rtimes P$ with P a p-group and C cyclic of order coprime to p; we identify P with a Sylow subgroup of G.

Lemma 4.1. *Let* N *be a normal subgroup of a finite group* G*, let* η *be an irreducible character of* N*, and let* θ *be a complex irreducible constituent of* $\operatorname{Ind}_N^G\eta$*. Write* $\mathcal{G}_\eta=\operatorname{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})$*, and similarly for* $\mathcal{G}_\theta$*. Then*

$$\frac{[\mathbb{Q}(\eta):\mathbb{Q}]}{[\mathbb{Q}(\theta):\mathbb{Q}]}=\frac{\#\{\gamma\in\mathcal{G}_\eta\mid\langle\eta^\gamma,\operatorname{Res}_N\theta\rangle\ne0\}}{\#\{\gamma\in\mathcal{G}_\theta\mid\langle\operatorname{Ind}_N^G\eta,\theta^\gamma\rangle\ne0\}}.$$

In particular, if $\operatorname{Ind}_N^G\eta$ *is irreducible, then*

$$\frac{[\mathbb{Q}(\eta):\mathbb{Q}]}{[\mathbb{Q}(\theta):\mathbb{Q}]}=\#\{\gamma\in\mathcal{G}_\eta\mid\langle\eta^\gamma,\operatorname{Res}_N\theta\rangle\ne0\}.$$

Proof. The G-action on the characters of N commutes with the Galois action. Every Galois conjugate of θ is a constituent of $\operatorname{Ind}^G\eta^\gamma$ for some $\gamma\in\mathcal{G}_\eta$, and moreover the number of distinct Galois conjugates of θ in η^γ is independent of γ.

Also, the number of Galois conjugates of η in $\mathrm{Res}_N\,\theta^\gamma$ is independent of $\gamma \in \mathcal{G}_\theta$. So an inclusion–exclusion count gives

$$\#\mathcal{G}_\theta = \#\mathcal{G}_\eta \cdot \frac{\#\{\gamma \in \mathcal{G}_\theta \mid \langle \mathrm{Ind}_N^G\,\eta, \theta^\gamma\rangle \neq 0\}}{\#\{\gamma \in \mathcal{G}_\eta \mid \langle \eta^\gamma, \mathrm{Res}_N\,\theta\rangle \neq 0\}}.$$

□

Lemma 4.2. *Let η be an irreducible complex representation of G, with rational hull $\hat{\eta}$. Then*

$$\dim\hat{\eta} = \dim\eta \cdot m(\eta)\cdot[\mathbb{Q}(\eta):\mathbb{Q}].$$

Proof. The rational hull of η is given by

$$\hat{\eta} = m(\eta)\sum_{\gamma\in\mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})}\eta^\gamma,$$

whence the claim follows. □

Theorem 4.3. *Let $G = C \rtimes X$ with C cyclic of order coprime to $|X|$. Let τ be a complex irreducible character of G with rational hull $\rho = \hat{\tau}$, let π be a complex irreducible constituent of $\mathrm{Res}_X\,\tau$ with rational hull $\hat{\pi}$, ψ an irreducible constituent of $\mathrm{Res}_C\,\tau$ with rational hull $\hat{\psi}$, K_ψ the stabiliser of ψ under the X-action on $\mathrm{Irr}(C)$, and let ξ be a complex irreducible constituent of $\mathrm{Res}_{K_\psi}\,\pi$. Then*

$$\begin{aligned}\mu(\rho, \mathrm{Ind}_X^G\,\hat{\pi}) &= \\ &= \frac{m(\pi)}{m(\tau)}\langle\xi, \mathrm{Res}_{K_\psi}\,\pi\rangle\cdot\#\{\textit{Galois conjugates } \pi' \textit{ of } \pi \mid \langle \mathrm{Res}_{K_\psi}\,\pi', \xi\rangle \neq 0\}\\ &= \frac{\dim\hat{\psi}\dim\hat{\pi}}{\dim\rho}.\end{aligned}$$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker\rho\cap C)$. So $K = K_\psi$ is assumed to be the kernel of the X-action on C. Recall that ψ denotes a complex constituent of $\tau|_C$. In particular, $\tau = \mathrm{Ind}_{CK}^G\,\psi\xi$, as explained in [14, Part II, §8.2]. We have

$$\begin{aligned}\rho &= m(\tau)\sum_{\gamma\in\mathrm{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})}\tau^\gamma; &\quad \dim\rho = m(\tau)[\mathbb{Q}(\tau):\mathbb{Q}]\dim\tau,\\ \hat{\pi} &= m(\pi)\sum_{\gamma\in\mathrm{Gal}(\mathbb{Q}(\pi)/\mathbb{Q})}\pi^\gamma; &\quad \dim\hat{\pi} = m(\pi)[\mathbb{Q}(\pi):\mathbb{Q}]\dim\pi.\end{aligned}$$

Thus

$$\begin{aligned}\mu(\rho, \mathrm{Ind}_X^G\,\hat{\pi}) &= \tfrac{1}{m(\tau)}\langle\tau, \mathrm{Ind}_X^G\,\hat{\pi}\rangle = \tfrac{1}{m(\tau)}\langle\mathrm{Ind}_{CK}^G\,\psi\xi, \mathrm{Ind}_X^G\,\hat{\pi}\rangle\\ &= \tfrac{1}{m(\tau)}\langle\mathrm{Res}_X\,\mathrm{Ind}_{CK}^G\,\psi\xi, \hat{\pi}\rangle = \tfrac{1}{m(\tau)}\langle\mathrm{Ind}_K^X\,\xi, \hat{\pi}\rangle = \tfrac{1}{m(\tau)}\langle\xi, \mathrm{Res}_K\,\hat{\pi}\rangle,\end{aligned}$$

where the last line follows from Mackey's formula, noting that $CK\backslash G/X$ consists of one double coset, and that $CK \cap X = K$.

Next, X acts on the representations of K by conjugation, and there is a Clifford theory decomposition

$$\mathrm{Res}_K\,\pi = e\sum_{g\in X/\mathrm{Stab}_X\,\xi}\xi^g. \tag{4.4}$$

Recall that the constituents of $\hat\pi$ are Galois conjugates of π, and we select those whose restriction to K contains ξ:

$$\Omega = \bigl\{\gamma \in \mathrm{Gal}(\mathbb{Q}(\pi):\mathbb{Q}) \bigm| \langle \mathrm{Res}_K \pi^\gamma, \xi\rangle \neq 0\bigr\}.$$

The inner product $\langle \mathrm{Res}_K \pi^\gamma, \xi\rangle = \langle \mathrm{Res}_K \pi, \xi^{\gamma^{-1}}\rangle$ is the same (and equals e) for every $\gamma \in \Omega$, since $\xi^{\gamma^{-1}}$ is irreducible and so must be one of ξ^g in (4.4). So we have

$$\frac{1}{m(\tau)}\langle \xi, \mathrm{Res}_K \hat\pi\rangle = \frac{m(\pi)}{m(\tau)}|\Omega|\langle \xi, \mathrm{Res}_K \pi\rangle,$$

which proves the first equality.

It remains to show that

$$\frac{m(\pi)}{m(\tau)}|\Omega|\langle \xi, \mathrm{Res}_K \pi\rangle = \frac{\dim\hat\psi \dim\hat\pi}{\dim\rho}. \tag{4.5}$$

By comparing dimensions in (4.4), and since $\tau = \mathrm{Ind}_{CK}^G \psi\xi$, we see that

$$\langle \xi, \mathrm{Res}_K \pi\rangle = e = \frac{\dim\pi}{[X:\mathrm{Stab}_X \xi]\dim\xi} = \frac{[X:K]\dim\pi}{[X:\mathrm{Stab}_X \xi]\dim\tau} = \frac{[\mathrm{Stab}_X \xi : K]\dim\pi}{\dim\tau},$$

so

$$\mu(\rho, \mathrm{Ind}^G \hat\pi) = \frac{m(\pi)}{m(\tau)}|\Omega|\langle \xi, \mathrm{Res}_K \pi\rangle = |\Omega|\cdot[\mathrm{Stab}_X \xi : K]\frac{m(\pi)\dim\pi}{m(\tau)\dim\tau}.$$

Consider the two groups

$$\begin{aligned} H_1 &= \{\gamma \in \mathrm{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q}) \mid \langle (\psi\xi)^\gamma, \mathrm{Res}_{CK}\tau\rangle \neq 0\},\\ H_2 &= \{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \mid \langle \xi^\gamma, \mathrm{Res}_K \pi\rangle \neq 0\}.\end{aligned}$$

There is a natural projection $H_1 \twoheadrightarrow H_2$ given by the restriction of Galois action to $\mathbb{Q}(\xi)$, whose kernel consists of precisely those elements of $\mathrm{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q})$ that act trivially on ξ, and through the action of some $g \in X$ on ψ (this last condition is equivalent to the Galois element being in H_1). Thus, the kernel is isomorphic to the subgroup of G/CK that acts trivially on ξ, i.e. to $\mathrm{Stab}_X \xi/K$. We deduce that

$$\mu(\rho, \mathrm{Ind}^G \hat\pi) = |\Omega|\frac{|H_1|}{|H_2|}\frac{m(\pi)\dim\pi}{m(\tau)\dim\tau}.$$

Now, by applying Lemma 4.1 first to $CK \triangleleft G$ with $\theta = \tau$, $\eta = \psi\xi$, and then to $K \triangleleft X$ with $\theta = \pi$, $\eta = \xi$, we find that

$$|H_1| = \frac{[\mathbb{Q}(\xi):\mathbb{Q}][\mathbb{Q}(\psi):Q]}{[\mathbb{Q}(\tau):\mathbb{Q}]} \qquad \text{and} \qquad |H_2| = |\Omega|\frac{[\mathbb{Q}(\xi):\mathbb{Q}]}{[\mathbb{Q}(\pi):\mathbb{Q}]},$$

so that

$$\begin{aligned}\mu(\rho, \mathrm{Ind}^G \hat\pi) &= |\Omega|\cdot\frac{|H_1|}{|H_2|}\frac{m(\pi)\dim\pi}{m(\tau)\dim\tau}\\ &= |\Omega|\cdot\frac{[\mathbb{Q}(\xi):\mathbb{Q}]\cdot[\mathbb{Q}(\psi):\mathbb{Q}]/[\mathbb{Q}(\tau):\mathbb{Q}]}{|\Omega|[\mathbb{Q}(\xi):\mathbb{Q}]/[\mathbb{Q}(\pi):\mathbb{Q}]}\cdot\frac{m(\pi)\dim\pi}{m(\tau)\dim\tau}\\ &= \frac{[\mathbb{Q}(\psi):\mathbb{Q}]\cdot[\mathbb{Q}(\pi):\mathbb{Q}]m(\pi)\dim\pi}{[\mathbb{Q}(\tau):\mathbb{Q}]m(\tau)\dim\tau} = \frac{\dim\hat\psi\dim\hat\pi}{\dim\rho},\end{aligned}$$

where the last equality follows from Lemma 4.2. ∎

Theorem 4.6. *Let $G = C \rtimes P$ be p-quasi-elementary, let ρ be an irreducible rational representation of G. Let ψ be a complex irreducible constituent of $\operatorname{Res}_C \rho$ with rational hull $\hat{\psi}$, and let $\hat{\pi}$ be a rational irreducible constituent of $\operatorname{Res}_P \rho$ of minimal dimension. Denote by π a complex irreducible constituent of $\hat{\pi}$, by ξ a complex irreducible consitutent of $\pi|_{K_\psi}$, where $K_\psi \le P$ is the stabiliser in P of ψ, and by τ a complex irreducible constituent of ρ such that $\operatorname{Res}_P \tau$ contains π. Then*

$$\textit{order of } \rho \textit{ in } \mathrm{C}(G) = \mu(\rho, \operatorname{Ind}_P^G \hat{\pi}) = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}$$
$$= \frac{m(\pi)}{m(\tau)} \langle \xi, \operatorname{Res}_{K_\psi} \pi \rangle \cdot \#\{\textit{Galois conjugates } \pi' \textit{ of } \pi \mid \xi \subset \operatorname{Res}_{K_\psi} \pi'\}.$$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$ (see Lemma 2.1). Thus, $K = K_\psi$ is assumed to be the kernel of the P-action on C. Under this assumption, if $H \le G$ intersects C non-trivially, then

$$\langle \rho, \mathbb{C}[G/H] \rangle_G = \langle \operatorname{Res}_H \rho, \mathbf{1} \rangle_H = 0.$$

Write o for the order of ρ in $\mathrm{C}(G)$. By Theorem 2.5, we have

$$\begin{aligned} o \cdot \langle \rho, \rho \rangle &= \gcd_{H \le G} \langle \rho, \mathbb{C}[G/H] \rangle_G = \gcd_{H \le P} \langle \rho, \mathbb{C}[G/H] \rangle_G \\ &= \gcd_{H \le P} \langle \rho|_P, \mathbb{C}[P/H] \rangle_P. \end{aligned}$$

Because $\mathrm{C}(P) = 1$ by the Ritter-Segal theorem [12, 13], we can replace the permutation representations $\mathbb{C}[P/H]$ by all rational representations of P in the last term. This is clearly the same as just taking the rational irreducible constituents $\hat{\pi}_1, ..., \hat{\pi}_k$ of $\rho|_P$, so

$$o = \frac{1}{\langle \rho, \rho \rangle} \gcd_j \langle \rho|_P, \hat{\pi}_j \rangle = \gcd_j \frac{\langle \rho, \operatorname{Ind}_P^G \hat{\pi}_j \rangle}{\langle \rho, \rho \rangle} = \gcd_j \mu(\rho, \operatorname{Ind}_P^G \hat{\pi}_j). \tag{4.7}$$

The theorem will therefore follow from Theorem 4.3, once we show that the gcd may be replaced by the term corresponding to any $\hat{\pi}$ of minimal dimension. Now, by Theorem 4.3 and by Lemma 4.2,

$$\mu(\rho, \operatorname{Ind}_P^G \hat{\pi}_j) = \frac{\dim \hat{\psi} \dim \hat{\pi}_j}{\dim \rho} = \frac{\dim \hat{\psi}\, m(\pi_j) \dim \pi_j [\mathbb{Q}(\pi_j) : \mathbb{Q}]}{\dim \rho},$$

where π_j is a complex irreducible constituent of $\hat{\pi}_j$. We argue as in [17, §2]: if $p = 2$, then all the terms $m(\pi_j)$, $\dim \pi_j$, $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are powers of 2, so gcd and minimum are the same. If p is odd, then $m(\pi_j) = 1$, and moreover, either some $\pi_j = \mathbf{1}$, in which case the claim is clear, or else all $\dim \pi_j$ are powers of p, while all $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are $(p-1)$ times powers of p ([17, Lemma 2.1]), so again gcd and minimum are the same. $\square$

5. Examples: $\mathrm{GL}_2(\mathbb{F}_q)$, $\mathrm{PGL}_2(\mathbb{F}_q)$, $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{PSL}_2(\mathbb{F}_q)$

Theorem 5.1. *For every prime power $q = p^n$, the group $G = \mathrm{GL}_2(\mathbb{F}_q)$ has $\hat{\mathrm{C}}(G) = \{1\}$.*

Proof. By Corollary 3.3, it suffices to show that every maximal quasi-elementary subgroup $Q = C \rtimes P$ of $G = \mathrm{GL}_2(\mathbb{F}_q)$ is contained in some $\bar{Q} < G$ with $\hat{\mathrm{C}}(\bar{Q}) = 1$. Pick $C = \langle g \rangle$ cyclic, and let $P = \mathrm{Syl}_l(N_G(C))$ for some prime number l. Write $f(t)$ for the characteristic polynomial of g.

Case 1 (split Cartan). Suppose $f(t)$ has distinct roots $a, b \in \mathbb{F}_q^\times$. Then g is conjugate to $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, and its centraliser is the split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times, \qquad N_G(C) < (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \rtimes C_2,$$

with $C_2 = \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$. Here $\bar{Q} = (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \rtimes C_2$ has trivial $\hat{\mathrm{C}}(\bar{Q})$ by Corollary 2.6.

Case 2 (non-split Cartan). Suppose $f(t)$ is irreducible over $\mathbb{F}_q$. Then the centraliser of C is the non-split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q[g]^\times \cong \mathbb{F}_{q^2}^\times, \qquad N_G(C) < \mathbb{F}_{q^2}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times \rtimes C_2.$$

Again $\bar{Q} = \mathbb{F}_{q^2}^\times \rtimes C_2$ has trivial $\hat{\mathrm{C}}$ by Corollary 2.6.

Case 3 (scalars). Suppose $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is a scalar matrix. Then $Q = C \rtimes P$ can be embedded into one of the following:

- if $l = p$: $\bar{Q} = C \times U = C \times \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$; in this case U is an elementary abelian p-group; or
- if either l is odd and $l|(q-1)$, or $l = 2$ and $q \equiv 1 \mod 4$: $\bar{Q} = H \rtimes C_2$ with $H =$ split Cartan; or
- if either l is odd and $l|(q+1)$, or $l = 2$ and $q \equiv 3 \mod 4$: $\bar{Q} = H \rtimes C_2$ with $H =$ non-split Cartan.

In all these cases, $\hat{\mathrm{C}}(\bar{Q})$ is trivial by Corollary 2.6.

Case 4 (non-semisimple). Finally suppose that g is not semisimple, say $g = g_s g_u$ with g_s central and $g_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ non-trivial unipotent. Then

$$\begin{aligned} N_G C = N_G \langle g_u \rangle &= \left\{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \,\middle|\, ac^{-1} \in \mathbb{F}_p^\times \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) | a \in \mathbb{F}_q^\times \right\} \cdot \left\{ \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) | c \in \mathbb{F}_q \right\} \cdot \left\{ \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right) | b \in \mathbb{F}_p^\times \right\} \\ &\cong \mathbb{F}_q^\times \times (\mathbb{F}_q \rtimes \mathbb{F}_p^\times). \end{aligned}$$

If $l = p$, then Q can be embedded into $\bar{Q} = \langle g_s \rangle \times U$, where $U = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \cong \mathbb{F}_q$ is an elementary abelian p-group. In this case $\hat{\mathrm{C}}(\bar{Q})$ is trivial by Corollary 2.6. Otherwise, Q can be embedded into $\bar{Q} \cong (\mathbb{F}_q^\times \times \langle g_u \rangle) \rtimes \mathbb{F}_p^\times$, where the action in the semi-direct product is faithful. If τ is an irreducible character of $\bar{Q}$ such that $\mathrm{Res}_{\langle g_u \rangle} \tau$ is faithful, then $\bar{Q}/\ker \tau$ satisfies the assumptions of Proposition 2.7 with $K = \{1\}$, so $\mathrm{Tr}\, \tau \in \mathrm{Perm}(\bar{Q})$. Otherwise, $\mathrm{Res}_{\langle g_u \rangle} \tau = \dim \tau \cdot \mathbf{1}$, so τ factors through an abelian quotient, and $\mathrm{Tr}\, \tau \in \mathrm{Perm}(\bar{Q})$ e.g. by Corollary 2.6. □

Remark 5.2. It is also not hard to deduce the structure of $\hat{\mathrm{C}}$ for the related classical groups:

- $G = \mathrm{PGL}_2(\mathbb{F}_q)$. Combined with Lemma 2.1, the theorem implies $\hat{\mathrm{C}}(G) = 1$.
- $G = \mathrm{SL}_2(\mathbb{F}_q)$. In general, $\hat{\mathrm{C}}(G) \neq 1$. For example, $\mathrm{SL}_2(\mathbb{F}_3)$ has $\mathrm{C} = 1$ and $\hat{\mathrm{C}} \cong \mathbb{Z}/2\mathbb{Z}$ (it has a 2-dimensional irreducible symplectic representation), and $\mathrm{SL}_2(\mathbb{F}_{17})$ has $\mathrm{C} \cong \mathbb{Z}/4\mathbb{Z}$.
- $G = \mathrm{PSL}_2(\mathbb{F}_q)$. It is a result of Solomon, announced in [15], that $\hat{\mathrm{C}}(G) = 1$. This can also be seen following the argument for GL_2 in Theorem 5.1: the analogues of $\bar{Q}$ are the images of $\bar{Q} \cap \mathrm{SL}_2(\mathbb{F}_q)$ in $\mathrm{PSL}_2(\mathbb{F}_q)$, and they are dihedral in Cases 1 and 2 of the theorem, elementary abelian or dihedral ($p = 2$) in Case 3 and isomorphic to $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$ in Case 4. Again, all these groups have $\hat{\mathrm{C}} = 1$, so $\hat{\mathrm{C}}(G) = 1$.

6. $\mathrm{PSL}_n(\mathbb{F}_p)$

Let ord_2 denote the 2-adic valuation of a rational number, $\mathrm{ord}_2\left(2^x \cdot \frac{a}{b}\right) = x$, where $2 \nmid ab$.

Theorem 6.1. *Let $k \geq 4$ be an integer, and p a prime. The groups $\mathrm{PSL}_k(\mathbb{F}_p)$, and therefore also $\mathrm{SL}_k(\mathbb{F}_p)$, have $\hat{\mathrm{C}}(G)$ of exponent divisible by $2^{\min(\mathrm{ord}_2(k),\mathrm{ord}_2(p-1))}$.*

In the remainder of the section we prove the theorem using Corollary 3.10. We will construct a 2-quasi-elementary subgroup $Q = C \rtimes P$ of $G = \mathrm{PSL}_k(\mathbb{F}_p)$ and a rational character ρ of Q such that $\mathrm{Ind}_Q^G \rho$ has order divisible by $2^{\min(\mathrm{ord}_2(k),\mathrm{ord}_2(p-1))}$ in $\hat{\mathrm{C}}(G)$.

Lemma 6.2. *Let p be an odd prime and $k \geq 4$ an integer. If $k = 4$, assume that $p \equiv 1 \pmod 4$. Then there exists a prime number l that divides $p^{k-2} - 1$ but does not divide $p^s - 1$ for any $s < k - 2$.*

Proof. This is a special case of Zsigmondy's Theorem [18]. □

Write Q_{2^N} for the generalised quaternion group of order 2^N.

Lemma 6.3. *The group $\mathrm{SL}_2(\mathbb{F}_q)$, $q = p^k$ has a 2-Sylow subgroup of the form*

- $S = \{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\} \cong C_p^k$ *if $p = 2$;*
- $S = \langle c, h\rangle \cong Q_{2^N}, c = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right), h = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ *with $\alpha \in \mathbb{F}_q^\times$ of exact order $2^{N-1} || q-1$, if $q \equiv 1 \mod 4$;*
- $S = \langle c, h\rangle \cong Q_{2^N}, c = \left(\begin{smallmatrix}\alpha & -\beta\\ \beta & \alpha\end{smallmatrix}\right), h = \left(\begin{smallmatrix}\gamma & \delta\\ \delta & -\gamma\end{smallmatrix}\right)$ *with $\alpha + \beta\sqrt{-1} \in \mathbb{F}_{q^2}^\times$ of exact order $2^{N-1} || q+1$ and any choice of $\gamma, \delta \in \mathbb{F}_q$ with $\gamma^2 + \delta^2 = -1$, if $q \equiv 3 \mod 4$.*

Conjugation by the matrix $\iota = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ is an automorphism of S, acting as -1 in the first case, as $c \mapsto c, h \mapsto h^{-1}$ in the second case, and as $c \mapsto c^{-1}, h \mapsto hc^{2m+1}$ for some m in the last case.

Proof. Direct computation. □

From now on, G will denote $\mathrm{PSL}_k(\mathbb{F}_p)$. The theorem only has content when k is even and p is odd, so we will assume this. Write

$$n = \mathrm{ord}_2(p-1) \geq 1, \qquad N = \mathrm{ord}_2(p^{k-2} - 1) \geq 3, \qquad m = \mathrm{ord}_2(k-2) \geq 1.$$

Case A: Either $k > 4$ or $p \equiv 1 \mod 4$. Let A be a generator of a non-split Cartan subgroup $\mathbb{F}_{p^{k-2}}^\times = \mathrm{GL}_1(\mathbb{F}_{p^{k-2}}) < \mathrm{GL}_{k-2}(\mathbb{F}_p)$, and l a prime divisor of $p^{k-2} - 1$ as in Lemma 6.2. The conditions on l imply that the normaliser of $\langle A^{\frac{p^{k-2}-1}{l}}\rangle \cong C_l$ in $\mathrm{GL}_{k-2}(\mathbb{F}_p)$ is generated by A and by the Frobenius automorphism $F \in \mathrm{Gal}(\mathbb{F}_{p^{k-2}}/\mathbb{F}_p)$ of order $k-2$. Note that F has determinant -1, since it is an odd permutation on a normal basis of $\mathbb{F}_{p^{k-2}}/\mathbb{F}_p$. Define

$$c_p = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & I_{k-2} \end{pmatrix}, \qquad c_l = \begin{pmatrix} 1 & & \\ & 1 & \\ & & A^{\frac{p^{k-2}-1}{l}} \end{pmatrix},$$

$$x = \begin{pmatrix} d^{-1} & & \\ & 1 & \\ & & U \end{pmatrix}, \qquad f = \begin{pmatrix} -1 & & \\ & 1 & \\ & & F^{(k-2)/2^m} \end{pmatrix},$$

where $U = A^{\frac{p^{k-2}-1}{2^N}}$ and $d = \det U$. We view these matrices as representing elements of $G = \mathrm{PSL}_k(\mathbb{F}_p)$. Write

$$C = \langle c_p c_l \rangle \cong C_{pl}, \quad P = \langle x, f \rangle \cong C_{2^N} \rtimes C_{2^m}, \quad Q = CP \cong (C_p \times C_l) \rtimes (C_{2^N} \rtimes C_{2^m}).$$

Note that C_{2^N} acts trivially on C_l, and through a C_{2^n} quotient on C_p, while C_{2^m} acts through a C_2 quotient on C_p and faithfully on C_l.

Case B: $p \equiv 3 \mod 4$ and $k = 4$. We take the same c_p as in Case A, and $C = \langle c_p \rangle$. A 2-Sylow of the centraliser of C in G is isomorphic to $\{1\} \times \mathrm{Syl}_2(\mathrm{SL}_2(\mathbb{F}_p))$, which is isomorphic to Q_{2^N} by the last case of Lemma 6.3. A 2-Sylow of the normaliser is

$$P = \mathrm{Syl}_2 N_G(C) = \mathrm{Syl}_2 Z_G(C) \rtimes \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \cong Q_{2^N} \rtimes \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

which is in fact isomorphic to the semi-dihedral group $SD_{2^{N+1}}$. Again, we let $Q = CP$.

In both cases, write K for the centraliser of C in P. Thus, $K \cong C_{2^{N-n}}$ in case A, and $K \cong Q_{2^N}$ in case B, where the isomorphism is that of Lemma 6.3. Let A_p be K in case A, and a cyclic subgroup of index 2 in K that is normal in Q in case B. Let χ be faithful irreducible characters of CA_p, $\tau = \mathrm{Ind}_{CA_p}^{Q} \chi$ and $\rho = \mathrm{Tr}\,\tau \in \mathrm{Irr}_{\mathbb{Q}}(Q)$.

Lemma 6.4. *The character ρ has order 2^n in $\hat{\mathrm{C}}(Q)$.*

Proof. We will use Proposition 2.7. The biggest subgroup of P that intersects CA_p trivially is of order 2 in case B, and of order 2^m in case A. So the order of ρ in $\hat{\mathrm{C}}(G)$ is $2^{N+1-(N-1)-1} = 2$ in case B, and $2^{N+m-(N-n)-m} = 2^n$ in case A. □

Finally, we show that $\mathrm{Ind}_Q^G \rho$ has order divisible by $2^{\min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))}$ in $\hat{\mathrm{C}}(G)$. We will use Corollary 3.10 with $Q_1 = Q_2 = Q$ and $\chi_1 = \chi_2 = \chi$. In view of Lemma 6.4, it suffices to show that

$$\sum_{[U] \in CC_{\mathrm{cyc}}(G)} S(U) \tag{6.5}$$

has 2-adic valuation at most $n - \min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))$, where for $U \leq CA_p$,

$$S(U) = \frac{[\mathbb{Q}(\tau):\mathbb{Q}]}{|CA_p|^2} |N_G(U)| \phi(|U|) \cdot \Bigg(\sum_{\substack{D \leq CA_p \\ D \sim U}} \frac{\mu([D : D \cap \ker \chi])}{\phi([D : D \cap \ker \chi])} \Bigg)^2.$$

Note that since CA_p is cyclic and χ is faithful, this simplifies to

$$S(U) = \frac{[\mathbb{Q}(\tau):\mathbb{Q}]}{|CA_p|^2 \phi(|U|)} |N_G(U)| \mu(|U|)^2,$$

see Remark 3.11. In particular, $S(U) = 0$ if U has non-square-free order.

Case A.

The subgroups of CK of square-free order are C_{2lp}, C_{lp}, C_{2l}, C_l, C_{2p}, C_p, C_2, and C_1. We will show that $S(C_{lp}) + S(C_{2lp})$ has a strictly lower 2-adic valuation than the rest of the sum, and that this valuation is $n - \min(\mathrm{ord}_2(k), n)$. A summary of the calculations that follow is:

$$\begin{aligned}
\mathrm{ord}_2[\mathbb{Q}(\tau):\mathbb{Q}] &= \mathrm{ord}_2(l-1)+N-n-1-m,\\
\mathrm{ord}_2\,|CK|^2 &= 2(N-n),\\
\phi(|C_{lp}|)=\phi(|C_{2lp}|) &= (l-1)(p-1),\\
|N_G(C_{lp})|=|N_G(C_{2lp})| &= \frac{(k-2)p(p^{k-2}-1)(p-1)}{\gcd(k,p-1)},\\
\mathrm{ord}_2(S(C_{lp})+S(C_{2lp})) &= \mathrm{ord}_2(2S(C_{lp}))\\
&= 1+\mathrm{ord}_2(l-1)+N-n-1-m-2(N-2)+N+\\
&\quad\; n+m-\min(\mathrm{ord}_2(k),n)-\mathrm{ord}_2(l-1)+n\\
&= n-\min(\mathrm{ord}_2(k),n).
\end{aligned}$$

The assertions concerning $|CK|$ and $\phi(|C_{lp}|)$ are clear.

Since the conjugation action of P on $\mathrm{Irr}(CK)$ is through Galois automorphisms, and $\ker(P\to \mathrm{Aut}(CK))$ has index 2^{n+m} in P, we have

$$[\mathbb{Q}(\tau):\mathbb{Q}] = 2^{-n-m}[\mathbb{Q}(\chi):\mathbb{Q}] = \frac{p-1}{2^n}\frac{(l-1)2^{N-n-1}}{2^m},$$

with 2-adic valuation $\mathrm{ord}_2(l-1)+N-n-1-m$.

The normaliser $N_{\mathrm{GL}_k(p)}$ of the preimage of C_{lp} under $\mathrm{SL}\to\mathrm{PSL}$ consists of block diagonal matrices, with the normaliser of non-split Cartan in the lower right corner (order $(k-2)(p^{k-2}-1)$), and a Borel subgroup in the top left (order $p(p-1)^2$). The determinant is surjective on $N_{\mathrm{GL}_k(p)}$, and $N_{\mathrm{GL}_k(p)}$ contains $Z(\mathrm{GL}_k(p))$, so the normaliser of C_{lp} in PSL has order $\frac{(k-2)(p^{k-2}-1)p(p-1)}{\gcd(k,p-1)}$, with 2-adic valuation $N+n+m-\min(\mathrm{ord}_2(k),n)$. This is also the normaliser of C_{2lp}.

It remains to show that the rest of the sum in equation (6.5) has strictly greater 2-adic valuation than $\mathrm{ord}_2(S(C_{lp})+S(C_{2lp}))$. If $U\le C$, then $|N_G(U)|$ and $|N_G(UC_2)|$ agree up to a power of p, $\phi(|U|)=\phi(|UC_2|)$, while $\mu(|U|)=-\mu(|UC_2|)$. It follows that the 2-adic valuation of $S(U)+S(UC_2)$ is at least 1 greater than that of $S(U)$.

Moreover, for any $U\le C_{lp}$, the normaliser of U in G contains that of C_{lp}, while $1/\phi(|U|)$ has strictly greater 2-adic valuation than $1/\phi(|C_{lp}|)$ whenever $U\ne C_{lp}$. This establishes the claim.

Case B. The subgroups of CA_p of square-free order are C_1, C_2, C_p, and C_{2p}. We will show that $\mathrm{ord}_2(\sum S(U)) = \mathrm{ord}_2(S(C_p)+S(C_{2p}))=0$. Again, we summarise the calculations as follows:

$$\begin{aligned}
\mathrm{ord}_2[\mathbb{Q}(\tau):\mathbb{Q}] &= N-3,\\
\mathrm{ord}_2\,|CA_p|^2 &= 2N-2,\\
\phi(|C_p|)=\phi(|C_{2p}|) &= p-1,\\
|N_G(C_p)|=p^4|N_G(C_{2p})| &= p^4\cdot\frac{(p-1)^3p^2(p+1)}{2},\\
\mathrm{ord}_2(S(C_p)+S(C_{2p})) &= \mathrm{ord}_2((1+p^4)S(C_{2p}))\\
&= 1+N-3-2N+2-1+N+1=0.
\end{aligned}$$

The assertions concerning $|CA_p|$ and ϕ are clear.

It follows from the description of the P-action on $\mathrm{Irr}(CK)$ that $[\mathbb{Q}(\tau):\mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(\chi):\mathbb{Q}]$, and has 2-adic valuation 2^{N-3}.

The normaliser of C_{2p} in GL_4 is block diagonal, with all invertible matrices in the bottom right corner, and Borel in the top left. So its order in PSL is $\frac{(p-1)^3p^2(p+1)}{2}$ with 2-adic valuation $N+1$. Finally, $|N(C_p)| = p^4|N(C_{2p})|$, e.g. see Murray [10] §4.

It remains to show that the 2-adic valuation of $S(C_1)+S(C_2)$ is positive. The normaliser of C_2 in GL_4 is $\mathrm{GL}_2\times\mathrm{GL}_2$, so the order of the normaliser in PSL is $\frac{(p-1)^3p^2(p+1)^2}{2}$, with 2-adic valuation $2N$, and the normaliser of C_1 is even bigger. So the 2-adic valuations of $S(C_1)$ and of $S(C_2)$ are positive.

Corollary 6.6. *As G ranges over the simple groups $\mathrm{PSL}_k(\mathbb{F}_p)$, and therefore also over $\mathrm{SL}_k(\mathbb{F}_p)$, the exponent of $\mathrm{C}(G)_2$ is unbounded.*

Proof. If $\mathrm{ord}_2(k) > \mathrm{ord}_2(p-1)$, then by [16, Lemma 5.6(1)] all Schur indices in $\mathrm{PSL}_k(\mathbb{F}_p)$ are trivial. So the assertion follows from Theorem 6.1. □

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